Alma Mater Studiorum · Universitá di Bologna

SCUOLA DI SCIENZE
Corso di Laurea Magistrale in Matematica

Schatten Classes, Interpolation Spaces, Singular Integral Operators and their Commutators

Tesi di Laurea Magistrale in Analisi Matematica

Relatore:
Chiar.ma Prof.ssa
Loredana Lanzani

Correlatore:
Chiar.mo Prof.
Tuomas P. Hytönen

Presentata da:
Lorenzo Zacchini

Anno Accademico 2023-2024

Contents

Introduction

As suggested by the title, this thesis covers various areas in mathematical analysis: it contains foundational topics in operator theory, interpolation spaces, and singular integrals, which provide a comprehensive framework for understanding the advanced results in the article by S. Janson and T. H. Wolff [14]. The contents of this significant work are discussed in Chapter 4. Below, I will summarize the contents of this thesis in greater detail.

The first Chapter aims to introduce the so-called Schatten classes, which are specific classes of compact operators on Hilbert spaces characterized by the fundamental property that the sequence of their singular values belongs to a certain ℓ^p space. To achieve this, several foundational topics are presented first, including compact operators with some of their basic properties, other classes of operators (normal, unitary, positive), the concept of diagonalization for compact self-adjoint operators, and finally the singular value decomposition of a compact operator on Hilbert spaces.

Chapter 2, on the other hand, aims to provide the foundational knowledge on interpolation spaces necessary for understanding the article [14]. Its contents are mainly drawn from the book [3]. In particular, after introducing the Riesz-Thorin theorem and the Marcinkiewicz theorem, two classical results, a general discussion of interpolation space theory is presented, with a specific focus on the Real Method. Subsequently, the specific case of interpolation using the Real Method for Lorentz spaces $L^{p,q}$ and Schatten-Lorentz classes $S^{p,q}$ is illustrated, where Lebesgue spaces L^p and Schatten classes S^p respectively appear as special cases.

The third Chapter of this work pursues multiple goals. The first is to provide the theoretical foundations for homogeneous singular integral operators of convolution type, so as to give context to the discussion in Chapter 4 about the article [14]. To this end, Section 3.1 presents sufficient conditions for these operators to be bounded from L^p to

L^p, for $1 < p < \infty$, and from L^1 to $L^{1,\infty}$. Two classic examples, the Hilbert and Riesz transforms, are also mentioned.

After stating these general results, the remainder of Chapter 3 focuses on the Hilbert transform in its various forms: on the line, on the circle, and on an arbitrary Lipschitz curve. This part was motivated by an interest in exploring this topic for its own sake, rather than as auxiliary results.

Sections 3.2 and 3.3 are developed in parallel to one another, to better highlight similarities and differences between the Hilbert transform on the line and that on the circle. A nearly identical analogy is that both are bounded from L^p to L^p for $1 < p < \infty$, and from L^1 to $L^{1,\infty}$. Moreover, in both cases, there exists an integral operator that, under suitable assumptions on the (real valued) function f to which it is applied, has as its real part the convolution of f with the Poisson kernel and as its imaginary part the convolution of f with the so-called conjugate Poisson kernel. Additionally, under appropriate assumptions, it is possible to show that such an integral converges to $f + iHf$ on the boundary, where for brevity we use the same notation H to denote the Hilbert transform on the line and on the circle. What sets the two situations (line and circle) apart from one another, however, are primarily the required assumptions on f, which differ in the two cases, as well as the integrals that are involved: twice the Cauchy integral for the line and the Herglotz integral for the circle.

In Section 3.4, the Hilbert transform on an arbitrary Lipschitz curve is discussed. The study of this case is particularly interesting because, unlike the cases of smoother curves, it cannot be reduced to the example of a straight line or a circle, thus requiring new techniques.

In Subsection 3.4.1, after introducing its definition, we give some glimpses of the proof of the L^2-boundedness of the Hilbert transform on Lipschitz curves, as presented in the important paper [7] by R. Coifman, A. McIntosh, and Y. Meyer. This work, published in 1982, improves upon an earlier, highly significant result by A. P. Calderón [4] from 1977, where the L^2-boundedness of the Hilbert transform on Lipschitz curves was proven, though with the restriction that the Lipschitz constant of the curve must be sufficiently small.

In Subsection 3.4.2, a property known as the *Plemelj formulae* is stated, which in some sense generalizes the case of the real line for the boundary values of Cauchy-type integrals.

The results presented in Chapter 3 are, for simplicity, all discussed in the specific case of $\mathbb{R}^n$, equipped with the n-dimensional Lebesgue measure, and with the $(n-1)$-dimensional Hausdorff measure in the case of surfaces. However, many of these results also hold for more general spaces and measures.

Finally, as mentioned at the beginning of this introduction, Chapter 4 contains a detailed presentation of the work in [14]. The main result proven in this paper is as follows. Suppose that T is a homogeneous singular integral operator of convolution type with kernel $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$, and, if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, denote by M_f the pointwise multiplication by f. Consider the commutator $C_f = M_f T - T M_f$. Then, necessary and sufficient conditions for $C_f \in S^p$ are that f belongs to an appropriate Besov space if $p > n$, or that f is constant if $0 < p \leq n$.

In Section 4.3, the proof of sufficiency is presented. Particularly noteworthy is the proof that, in the case $p > n$, if f belongs to an appropriate Besov space, then C_f is in the Schatten class S^p. This proof utilizes a result concerning mixed norm spaces $L^p(L^q)$ and several results from interpolation space theory. Moreover, while following the general structure of the proof in [14], the version presented here includes some slight differences: the version proposed in this thesis does not require knowledge of results related to interpolation of Besov spaces using the real method.

In Section 4.4, the proof of the reverse implication for the case $p > n$ is presented: if C_f belongs to the Schatten class S^p, then f belongs to a specific Besov space. This proof is highly technical, and its core idea is to estimate the mean oscillation, first over single dyadic cubes and then by averaging over all dyadic partitions of $\mathbb{R}^n$. Through this approach, the condition for f to belong to the desired Besov space is obtained.

Bibliography

[1] L. V. Ahlfors, *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*, Third Edition, McGraw-Hill, 1978.

[2] S. R. Bell, *The Cauchy Transform, Potential Theory and Conformal Mapping*, Second Edition, CRC Press, 2015.

[3] J. Bergh, J. Löfström, *Interpolation spaces, an introduction*, Springer, 1976.

[4] A. P. Calderón, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci., Vol. 74, pp. 1324-1327, 1977.

[5] R. E. Castillo, H. C. Chaparro, *Classical and Multidimensional Lorentz Spaces*, De Gruyter, 2021.

[6] R. E. Castillo, H. Rafeiro, *An Introductory Course in Lebesgue Spaces*, Springer, 2016.

[7] R. Coifman, A. McIntosh, Y. Meyer, *The Hilbert transform on Lipschitz curves*, Proc. Centre Math. Anal. Austral. Nat. Univ., Vol. 1, pp. 26-69, 1982.

[8] L. C. Evans, R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, 1992.

[9] T. W. Gamelin, *Complex Analysis*, Springer, 2001.

[10] S. R. Garcia, J. Mashreghi, W. T. Ross, *Operator Theory by Example*, Oxford Univ. Press, 2023.

[11] A. Gogolin, *Lectures on Complex Integration*, Springer, 2014.

[12] L. Grafakos, *Classical Fourier Analysis*, Third Edition, Springer, 2014.

[13] T. Hytönen, J. van Neerven, M. Veraar, L. Weis, *Analysis in Banach Spaces, Volume I, Martingales and Littlewood-Paley Theory*, Springer, 2016.

[14] S. Janson, T. H. Wolff, *Schatten classes and commutators of singular integral operators*, Ark. Mat., Vol. 20(2), pp. 301-310, 1982.

[15] F. Jones, *Lebesgue Integration on Euclidean Space*, Revised Edition, Jones and Bartlett Publishers, 2001.

[16] Y. Katznelson, *An Introduction to Harmonic Analysis*, Third Edition, Cambridge University Press, 2004.

[17] J. D. McGovern, *The Hilbert Transform*, MS Thesis, McMaster University, Hamilton, Ontario, 1980. https://macsphere.mcmaster.ca/handle/11375/19310

[18] R. Remmert, *Theory of Complex Functions*, Springer, 1991.

[19] W. Rudin, *Functional analysis*, Second Edition, McGraw-Hill, 1991.

[20] W. Rudin, *Real and Complex Analysis*, Third Edition, McGraw-Hill, 1987.

[21] B. Russo, *On the Hausdorff-Young theorem for integral operators*, Pac. J. Math., Vol. 68, pp. 241-253, 1977.

[22] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970.

[23] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, 1986.

[24] V. Turunen, *Functional Analysis*, Course Lecture Notes, Aalto University, 2021. https://mycourses.aalto.fi/mod/resource/view.php?id=978164

[25] J. Verdera, *L^2 Boundedness of the Cauchy integral and Menger curvature*, Contemp. Math., Vol. 277, pp. 139-158, 2001.